# ON A CLASS OF FINITE DIMENSIONAL CONTRACTIVE PERTURBATIONS OF RESTRICTED SHIFTS OF FINITE MULTIPLICITY†

BY

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ABSTRACT

We study a class of finite-dimensional contractive perturbations of shift operators of finite multiplicity restricted to left invariant subspaces of vectorial $H^2$ spaces. We determine their spectra in terms of the characteristic function of the unperturbed operator and the perturbation.

## 1. Introduction

D. N. Clark studied one-dimensional perturbations of shift operators in $H^2$ restricted to their *-invariant (left-invariant) subspaces, which are unitary (see [1]). In this paper we are interested mostly in special finite-dimensional contractive perturbations of restricted shift operators of higher multiplicity and the relation between the characteristic functions of the perturbed and unperturbed operators. This yields information about the perturbation. The results stated in this paper have applications to the theory of stability of linear control systems; these findings will be published elsewhere.

I want to thank D. N. Clark for a preprint of his paper [1] which motivated me and which was a great help throughout this work.

Let $N$ be a separable Hilbert space. $H^2(N)$ is the Hardy class of order 2, that is, the set of $N$-valued functions on the unit circle satisfying

$$\| F \|^2 = \frac{1}{2\pi} \int_0^{2\pi} \| F(e^{it}) \|^2 dt < \infty$$

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and whose Fourier coefficients vanish for all negative indices. The functions in $H^2(N)$ have analytic continuation into the unit disc from which they are recoverable almost everywhere as radial limits. We will refer to both the function, as defined on the unit circle, and to its analytic continuation by the same letter and will work with both representations simultaneously. For a comprehensive treatment of the $H^2(N)$ spaces we refer to [3] and [5]. In this paper we will consider *-invariant subspaces $M$ whose orthogonal complement has a representation $SH^2(N)$ where $S$ is an inner function [3]. Furthermore we will assume $S(0)$ to be purely contractive at the origin, that is, $I - S(0)^*S(0) > 0$. This in turn implies $I - S(0)S(0)^* > 0$.

If $N$ is finite dimensional, as we will generally assume, then $I - S(0)^*S(0)$ and $I - S(0)S(0)^*$ are unitarily equivalent.

We denote by $P_M$ the orthogonal projection of $H^2(N)$ onto $M$ and, if $H^2(N)$ is embedded in $L^2(N)$, then also the projection of $L^2(N)$ onto $M$. We define the operator $T$ in $M$ by

$$TF = P_M(zF), \text{ for all } F \in M. \tag{1}$$

It follows that

$$(T^*F)(z) = \frac{F(z) - F(0)}{z} \tag{2}$$

where $T^*$ is the restriction of the left shift in $H^2(N)$ to the *-invariant subspace $M$.

Given any $N$-operator valued analytic function $A(z)$ in the unit disk we define $\tilde{A}(z)$ by $\tilde{A}(z) = A(\bar{z})^*$. Note that $S$ is inner if and only if $\tilde{S}$ is inner. We define $\tilde{M}$ by $\tilde{M} = H^2(N) \ominus \tilde{S}H^2(N)$.

THEOREM 1.1. ([2].) *The operator $T$ in $M$ defined by* (1) *is unitarily equivalent to the left shift restricted to $\tilde{M}$.*

We note that the unitary map is given by $\tau: M \to \tilde{M}$ where $\tau$ is defined by

$$(\tau F)(e^{it}) = e^{-it}S(e^{-it})^*F(e^{-it}). \tag{3}$$

For further use we note that

$$(\tau^*F)(e^{it}) = e^{-it}\tilde{S}(e^{-it})^*F(e^{-it}) \tag{4}$$

$$\tau P_M = P_{\tilde{M}}\tau. \tag{5}$$

and

$$\tau T = \tilde{T}^*\tau \tag{6}$$

where $\tilde{T}F = P_{\tilde{M}}(zF)$ for all $F \in \tilde{M}$.

LEMMA 1.2. *Let $\xi \in N$; then we have the following two cases.*

(i) $P_M \xi = (I - S(z)S(0)^*)\xi$; *here $P_M\xi$ is the projection of the constant function $\xi$.*

(ii) $P_M e^{-it}S(e^{it})\xi = e^{-it}(S(e^{it}) - S(0))\xi$, *thus $(S(z) - S(0))\xi/z \in M$ for all $\xi \in N$.*

PROOF. (i) Write $\xi = (\xi - S(z)\eta) + S(z)\eta$, where we want $\xi - S(z)\eta \in M$. Hence for all $\zeta \in N$ we must have:

$$0 = \frac{1}{2\pi}\int_0^{2\pi}(S(e^{it})\zeta, \xi - S(e^{it})\eta)dt = (S(0)\zeta, \xi) - (\zeta, \eta)$$

$$= (\zeta, S(0)^*\xi - \eta)$$

which implies $\eta = S(0)^*\xi$. Here we have used the fact that $S(e^{it})$ is unitary almost everywhere, since $S$ is inner by assumption. The condition $\eta = S(0)^*\xi$ is sufficient for the above decomposition to hold, since for $n \geqq 1$ and all $\zeta \in N$

$$\frac{1}{2\pi}\int_0^{2\pi}((I - S(e^{it})S(0)^*)\xi, S(e^{it})e^{int}\zeta)$$

$$= (\xi, Sz^n\zeta) - (S(0)^*\xi, z^n\zeta) = 0.$$

(ii) If $\tau$ is the transformation defined by (3) then $\tau(e^{-it}S(e^{it})\xi) = \xi$. Therefore by (4) and (5)

$$P_M(e^{-it}S(e^{it})\xi) = \tau^* P_{\tilde{M}}\tau(e^{-it}S(e^{it})\xi)$$

$$= \tau^* P_{\tilde{M}}(\xi) = \tau^*(I - \tilde{S}(e^{it})S(0))\xi = e^{-it}(S(e^{it}) - S(0))\xi$$

and we have proved (ii).

Thus in $M$ we have distinguished two subspaces which are defined as follows.

$$k_0 = \{(I - S(z)S(0)^*)\xi \mid \xi \in N\}$$

and

$$K_0 = \left\{\frac{S(z) - S(0)}{z}\xi \mid \xi \in N\right\}.$$

Following Clark's approach we obtain Lemma 1.3.

LEMMA 1.3. *Let $F \in M$; then*

(i) $(T^*F)(z) = F(z)/z$ *if and only if* $F \perp k_0$;

(ii) $(TF)(z) = zF(z)$ *if and only if* $F \perp K_0$.

PROOF. (i) $(T^*F)(z) = F(z)/z$ if and only if $F(0) = 0$, which is equivalent to $(F, \xi) = 0$ (for all $\xi \in N$), the inner product taken in $L^2(N)$. However since $P_M F = F$ we get $(F, \xi) = (P_M F, \xi) = (F, P_M \xi)$. But $P_M \xi = (I - S(z)S(0)^*)\xi$ by Lemma 1.2 and hence the result.

(ii) We observe that the unitary map $\tau$ sends $k_0$ onto $\tilde{K}_0$ and $K_0$ onto $\tilde{k}_0$, the subspaces of $\tilde{M}$ which are defined by

$$\tilde{k}_0 = \{(I - \tilde{S}(z)S(0))\xi \mid \xi \in N\} \text{ and}$$

$$\tilde{K}_0 = \left\{\left(\frac{\tilde{S}(z) - \tilde{S}(0)}{z}\right)\xi \mid \xi \in N\right\}, \text{ for}$$

$$\begin{aligned}\tau(I - S(e^{it})S(0)^*)\xi &= e^{-it}S(e^{-it})^*(I - S(e^{-it})S(0)^*)\xi \\ &= e^{it}(\tilde{S}(e^{it}) - \tilde{S}(0))\xi\end{aligned}$$

and similarly

$$\begin{aligned}\tau(e^{-it}(S(e^{it}) - S(0))\xi) &= e^{-it}S(e^{-it})^* \ (e^{it}(S(e^{-it}) - S(0))\xi) \\ &= (I - \tilde{S}(e^{it})S(0))\xi = (I - \tilde{S}(e^{it})\tilde{S}(0)^*)\xi.\end{aligned}$$

Now $F \perp K_0$ if and only if $\tau F \perp \tilde{k}_0$ which by part (i) occurs if and only if $\tilde{T}^*(\tau F) = (\tau F)/z$. But as $\tau T = \tilde{T}^*\tau$ by (6) we obtain $TF = \tau^*((\tau F)/z) = zF$.

LEMMA 1.4. *Let $F \in M$; then*

(i) $$(P_{k_0}F)(z) = (I - S(z)S(0)^*)(I - S(0)S(0)^*)^{-1}F(0);$$

(ii) $$(P_{K_0}F)(z) = \left(\frac{S(z) - S(0)}{z}\right)(I - S(0)^*S(0))^{-1}(\tau F)(0).$$

PROOF.

(i) $$(P_{k_0 \perp}F)(z) = F(z) - (I - S(z)S(0)^*)\xi,$$

and we must have $F(0) - (I - S(0)S(0)^*)\xi = 0$.

(ii) Using the transformation $\tau$ we have $F(z) - (S(z) - S(0)/z)\xi \in K_0^\perp$ if and only if $(\tau F)(z) - (I - \tilde{S}(z)S(0))\xi \in \tilde{k}_0^\perp$ for which a necessary and sufficient condition is $\xi = (I - S(0)^*S(0))^{-1}(\tau F)(0)$.

LEMMA 1.5.

(i) $$\| (I - S(z)S(0)^*)\xi \|_M^2 = \| \xi \|^2 - \| S(0)^*\xi \|^2.$$

(ii) $$\left\| \left(\frac{S(z) - S(0)}{z}\right)\xi \right\|_M^2 = \| \xi \|^2 - \| S(0)\xi \|^2.$$

The proof is computational and is omitted.

LEMMA 1.6.

(i) $$T^*(I - S(z)S(0)^*)\xi = -\left(\frac{(S(z) - S(0)}{z}\right)S(0)^*\xi$$

(ii) $$T\left(\frac{(S(z) - S(0)}{z}\right)\xi = -(I - S(z)S(0)^*)S(0)\xi\,.$$

PROOF.

(i) $$T^*(I - S(z)S(0)^*)\xi = \frac{(I - S(z)S(0)^*)\xi - (I - S(0)S(0)^*)\xi}{z}.$$

$$= -\left(\frac{S(z) - S(0)}{z}\right)S(0)^*\xi.$$

(ii) $$T\left(\frac{(S(z) - S(0)}{z}\right)\xi = (\tau^*\tilde{T}^*\tau)\left(\frac{(S(z) - S(0)}{z}\right)\xi$$

$$= (\tau^*\tilde{T}^*)(I - \tilde{S}(z)\tilde{S}(0)^*)\xi = \tau^*\left\{-\left(\frac{(\tilde{S}(z) - \tilde{S}(0)}{z}\right)\tilde{S}(0)^*\xi\right\}$$

$$= -(I - S(z)S(0)^*)S(0)\xi.$$

Theorem 1.7 summarizes results needed in the following sections.

THEOREM 1.7. *Let $P$ and $Q$ be strictly positive unitarily equivalent operators in a Hilbert space $H$.*

(i) *$X$ is a solution of the inequality $P \geqq X^*PX$ if and only if $X = P^{-\frac{1}{2}}AP^{\frac{1}{2}}$ with $A$ a contraction in $H$.*

(ii) *$X$ is a solution of $P = X^*PX$ if and only if $X = P^{-\frac{1}{2}}UP^{\frac{1}{2}}$ with $U$ isometric in $H$.*

(iii) *$X$ is a solution of the system of inequalities $P \geqq X^*QX$ and $Q \geqq XPX^*$ if and only if $X$ is a contraction satisfying $XP = QX$.*

(iv) *$X$ is a solution of the system of equations $P = X^*QX$ and $Q = XPX^*$ if and only if $X$ is unitary and $QX = XP$.*

PROOF.

(i) If $X = P^{-\frac{1}{2}}AP^{\frac{1}{2}}$ and $A$ is a contraction, then

$$P - X^*PX = P - (P^{\frac{1}{2}}A^*P^{-\frac{1}{2}})P(P^{-\frac{1}{2}}AP^{\frac{1}{2}})$$

$$= P^{\frac{1}{2}}(I - A^*A)P^{\frac{1}{2}} \geqq 0.$$

Conversely, assume $X$ solves $P \geqq X^*PX$; then for each vector $x$ in $H$ we have $\| P^{\frac{1}{2}}|x| \geqq \| P^{\frac{1}{2}}Xx \|$. Define an operator $A$ in $H$ by $AP^{\frac{1}{2}}x = P^{\frac{1}{2}}Xx$. $A$ is obviously a well-defined contractive linear operator. Thus $X = P^{-\frac{1}{2}}AP^{\frac{1}{2}}$.

(ii) If $X = P^{-\frac{1}{2}}UP^{\frac{1}{2}}$ with $U$ unitary then

$$X^*PX = (P^{\frac{1}{2}}U^*P^{-\frac{1}{2}})P(P^{-\frac{1}{2}}UP^{\frac{1}{2}}) = P^{\frac{1}{2}}U^*UP^{\frac{1}{2}} = P.$$

Conversely, if $P = X^*PX$, we have $\| P^{\frac{1}{2}}Xx \| = \| P^{\frac{1}{2}}x \|$. We define $U$ by $UP^{\frac{1}{2}}x = P^{\frac{1}{2}}Xx$. Obviously $U$ is isometric and $X = P^{-\frac{1}{2}}UP^{\frac{1}{2}}$. If $H$ is finite dimensional then $U$ is unitary.

(iii) Let $X$ be a contraction which satisfies $XP = QX$. Then also $PX^* = X^*Q$ holds. Therefore

$$(X^*X)P = X^*(XP) = X^*(QX) = (X^*Q)X - (PX^*)X = P(X^*X).$$

Hence, by induction, $(X^*X)P^n = P^n(X^*X)$ and, more generally, $(X^*X)f(P) = f(P)\,(X^*X)$ for every polynomial $f$. An approximation argument based on the spectral theorem implies $P^{\frac{1}{2}}(X^*X) = (X^*X)P^{\frac{1}{2}}$. Therefore $X^*QX = X^*XP = P^{\frac{1}{2}}X^*XP^{\frac{1}{2}} \leqq P$. A similar argument yields the other inequality.

Now assume $X$ solves the system of inequalities $P \geqq X^*QX$ and $Q \geqq XPX^*$. Since $Q$ and $P$ are unitarily equivalent, by assumption, we have $Q = U^*PU$ for some unitary operator $U$. Hence $P \geqq X^*QX = X^*U^*PUX = (UX)^*P(UX)$. By part (i) we have $UX = P^{-\frac{1}{2}}CP^{\frac{1}{2}}$ where $C$ is a contraction, so $X = U^*P^{-\frac{1}{2}}CP^{\frac{1}{2}}$. But $UQ = PU$ implies $UQ^{\frac{1}{2}} = P^{\frac{1}{2}}U$; therefore it follows that $U^*P^{-\frac{1}{2}} = Q^{-\frac{1}{2}}U^*$ or $X = U^*P^{-\frac{1}{2}}CP^{\frac{1}{2}} = Q^{-\frac{1}{2}}(U^*C)P^{\frac{1}{2}} = Q^{-\frac{1}{2}}AP^{\frac{1}{2}}$ where $A$ is a contraction.

Arguing similarly, starting from the inequality $Q \geqq XPX^*$, we obtain $X = Q^{\frac{1}{2}}A_1P^{-\frac{1}{2}}$ where $A_1$ is a contraction.

Thus $Q^{\frac{1}{2}}A_1P^{-\frac{1}{2}} = Q^{-\frac{1}{2}}AP^{\frac{1}{2}}$ or $QA_1 = AP$ which implies $Q^{\frac{1}{2}}A_1 = AP^{\frac{1}{2}}$. Hence $X = Q^{\frac{1}{2}}A_1P^{-\frac{1}{2}} = AP^{\frac{1}{2}}P^{-\frac{1}{2}} = A$ and similarly $X = A_1$. Thus $X$ is a contraction and moreover it satisfies $XP = QX$.

(iv) If $U$ is unitary and $P = U^*QU$ then obviously $UP = QU$.

Conversely if $X$ solves $P = X^*QX$ and $Q = XPX^*$ we obtain $P = X^*QX = (X^*X)P(X^*X)$. By part (ii) $X^*X = P^{-\frac{1}{2}}VP^{\frac{1}{2}}$ with $V$ isometric. Since $X^*X$ is self adjoint we have $P^{\frac{1}{2}}V^*P^{-\frac{1}{2}} = P^{-\frac{1}{2}}VP^{\frac{1}{2}}$ or $PV^* = VP$ which implies $P^{\frac{1}{2}}V^* = VP^{\frac{1}{2}}$. Hence $X^*X = V^*$ and $X^*X = V$ by taking adjoints. Thus $V$ is a positive isometry and therefore we must have $V = I$. So $X^*X = I$ and $X$ is an isometry. From $P = X^*QX$ we get $PX^* = X^*Q$ which is equivalent to $XP = QX$. Starting from the second inequality and using the same argument we get the result that $X^*$ is isometric, thus $X$ is unitary.

## 2. The perturbations

We define the class of perturbations under study in this section as follows.

DEFINITION 2.1. Let $A$ be a bounded linear operator on $N$. The linear transformation $Z(A)$ in $M$ is defined by

$$(Z(A)F)(z) = \begin{cases} zF(z) \text{ if } F \perp K_0 \\ (I - S(z)S(0)^*)A\xi \text{ if } F(z) = \dfrac{(S(z) - S(0))}{z}\xi . \end{cases}$$

REMARK 2.2. It follows that $Z(A)^*$ is given by

$$(Z(A)^*F)(z) = \begin{cases} \dfrac{F(z)}{z} \text{ if } F \perp k_0 \\ \left(\dfrac{(S(z) - S(0)}{z}\right)A^*\xi \text{ if } F(z) = (I - S(z)S(0)^*)\xi. \end{cases}$$

We also note that $T = Z(-S(0))$.

THEOREM 2.3. *Let the Hilbert space $N$ be finite dimensional; then*

(i) *$Z(A)$ is a contraction if and only if $A$ is a contraction which satisfies*

$$AS(0)^*S(0) = S(0)S(0)^*A; \tag{7}$$

(ii) *$Z(A)$ is unitary if and only if $A$ is unitary and satisfies* (7).

PROOF. We note that $Z(A)$ maps $K_0^\perp$ into $k_0^\perp$ and $K_0$ into $k_0$. Since $Z(A)$ is isometric on $K_0^\perp$ it will be a contraction if and only if it is contractive on $K_0$. This amounts to

$$\| A\xi \|^2 - \| S(0)^*A\xi \|^2 \leqq \| \xi \|^2 - \| S(0)\xi \|^2 \text{ for all } \xi \in N.$$

Similar reasoning applied to $Z(A)^*$ results in the analogous inequality

$$\| A^*\xi \|^2 - \| S(0)A^*\xi \|^2 \leqq \| \xi \|^2 - \| S(0)^*\xi \|^2 \text{ for all } \xi \in N.$$

These two inequalities are easily seen to be equivalent to the following system of operator inequalities:

$$\begin{aligned} I - S(0)^*S(0) &\geqq A^*(I - S(0)S(0)^*)A \text{ and} \\ I - S(0)S(0)^* &\geqq A(I - S(0)^*S(0))A^*. \end{aligned} \tag{8}$$

Now since $N$ is finite dimensional $I - S(0)^*S(0)$ and $I - S(0)S(0)^*$ are unitarily equivalent and invertible by our assumption on $S$. Thus we can apply Theorem 1.7 to obtain the result. We note that (7) holds if and only if

$$A(I - S(0)^*S(0)^{\frac{1}{2}} = (I - S(0)S(0)^*)^{\frac{1}{2}}A \text{ and} \tag{9}$$

$$(I - S(0)^*S(0))^{\frac{1}{2}}A^* = A^*(I - S(0)S(0)^*)^{\frac{1}{2}} \text{ hold.} \tag{10}$$

COROLLARY 2.4. *If $A$ is contractive, that is, $I - A^*A \geqq 0$ and satisfies* (7), *then*

$$(I - S(0)^*S(0))^{\frac{1}{2}}A^*A = A^*A(I - S(0)^*S(0))^{\frac{1}{2}} \text{ and} \tag{11}$$

$$(I - S(0)^*S(0))^{\frac{1}{2}}(I - A^*A)^{\frac{1}{2}} = (I - A^*A)^{\frac{1}{2}}(I - S(0)^*S(0))^{\frac{1}{2}}. \tag{12}$$

PROOF. Follows from (9) and (10).

Similar relations can be obtained by duality.

## 3. Characteristic functions of perturbations

In this section we apply the Sz.-Nagy-Foias theory of characteristic functions for contractions in a Hilbert space to study properties of the family of perturbations $Z(A)$ defined in Definition 2.1. This is done by calculating the characteristic function of $Z(A)$.

We recall (see [5]) that the characteristic function of a contraction $T$ in a Hilbert space $H$ is a triple $\{\mathscr{D}_T, \mathscr{D}_{T^*}, \Theta_T(\lambda)\}$ where

$$D_T = (1 - T^*T)^{\frac{1}{2}}, \; D_{T^*} = (1 - TT^*)^{\frac{1}{2}} \text{ and}$$
$$\mathscr{D}_T = \overline{D_T H}, \; \mathscr{D}_{T^*} = \overline{D_{T^*}H} \text{ and}$$
$$\Theta_T(\lambda) = [-T + \lambda D_{T^*}(1 - \lambda T^*)^{-1}D_T] \,|\, \mathscr{D}_T.$$

$\mathscr{D}_T, \mathscr{D}_{T^*}$ are the defect spaces of $T$ and measure the difference of $T$ from being unitary. Since we have

$$TD_T = D_{T^*}T \tag{13}$$

$\Theta_T$ is an analytic operator-valued function whose values are contractive operators from $\mathscr{D}_T$ into $\mathscr{D}_{T^*}$.

LEMMA 3.1. *If $A$ is a strict contraction in $N$ satisfying* (7) *then $\mathscr{D}_{Z(A)} = K_0$. $\mathscr{D}_{Z(A)^*} = k_0$ and*

$$D_{Z(A)}\left(\left(\frac{S(z) - S(0)}{z}\right)\xi\right) = \left(\frac{S(z) - S(0)}{z}\right)(I - A^*A)^{\frac{1}{2}}\xi \tag{14}$$

$$D_{Z(A)^*}(I - S(z)S(0)^*)\xi = (I - S(z)S(0)^*)(I - AA^*)^{\frac{1}{2}}\xi \tag{15}$$

*for all* $\xi \in N$.

PROOF. From Definition 2.1 we have $Z(A) = W + R$ where $W$ is a partial isometry defined by $W = Z(A)P_{K_0}^{\perp}$ with initial space $K_0^{\perp}$ and

$$(RF)(z) = (I - S(z)S(0)^*)A(I - S(0)^*S(0))^{-1}(\tau F)(0).$$

Thus $F \perp K_0$ implies $(\tau F)(0) = 0$ or $RF = 0$. Since $W^* \mid k_0 = 0$ we obtain $W^*R = 0$ and therefore also $R^*W = 0$. It follows that

$$I - Z(A)^*Z(A) = I - W^*W - W^*R - R^*W - R^*R = I - W^*W - R^*R.$$

Since $W$ is a partial isometry with $K_0^{\perp}$ as initial space, $W^*W = P_{K_0^{\perp}}$ and $I - W^*W = P_{K_0}$. From the definition of $R$ we easily obtain

$$(R^*RF)(z) = \left(\frac{S(z) - S(0)}{z}\right)A^*A\xi \text{ for } F(z) = \frac{S(z) - S(0)}{z}\xi.$$

Combining the results we obtain

$$((I - Z(A)^*Z(A)F)(z) = \left(\frac{S(z) - S(0)}{z}\right)(I - A^*A)(I - S(0)^*S(0))^{-1}(\tau F)(0).$$

By approximation arguments used before we have

$$\begin{aligned} &((I - Z(A)^*Z(A))^{\frac{1}{2}}F)(z) \\ (16)\qquad &= \frac{S(z) - S(0)}{z}(I - A^*A)^{\frac{1}{2}}(I - S(0)^*S(0))^{-1}(\tau F)(0). \end{aligned}$$

As we assumed $A$ to be strictly contractive, $(I - A^*A)^{\frac{1}{2}}$ is onto $N$, hence $(I - Z(A)^*Z(A))^{\frac{1}{2}}$ is onto $K_0$, and thus $\mathscr{D}_{Z(A)} = K_0$ and (14) holds. The results for $D_{Z(A)^*}$ are similarly obtained.

Now we proceed with the calculation of the characteristic function of $Z(A)$. For this we have to calculate

$$(I - \lambda Z(A)^*)^{-1}\left(\frac{(S(z) - S(0)}{z}\right)\eta.$$

For $|\lambda| < 1$, $I - \lambda Z(A)^*$ is invertible. Let

$$(I - \lambda Z(A)^*)^{-1}\left(\frac{S(z) - S(0)}{z}\right)\eta = F(z).$$

Then

$$\left(\frac{S(z) - S(0)}{z}\right)\eta = (I - \lambda Z(A)^*)F(z)$$

$$= F(z) - \frac{\lambda}{z}\{F(z) - (I - S(z)S(0)^*)(I - S(0)S(0)^*)^{-1}F(0)\} -$$

$$- \lambda\left(\frac{S(z) - S(0)}{z}\right)A^*(I - S(0)S(0)^*)^{-1}F(0)$$

or

$$F(z) = \{(S(z) - S(0))[\eta + \lambda A^*(I - S(0)S(0)^*)^{-1}F(0)] -$$

$$(17) \qquad - \lambda(I - S(z)S(0)^*)(I - S(0)S(0)^*)^{-1}F(0)\}/z - \lambda.$$

Since $F$ is analytic in the open unit disc, the numerator vanishes at $z = \lambda$. Thus we obtain

$$F(0) = (I - S(0)S(0)^*)[(I - S(\lambda)S(0)^*) - (S(\lambda) - S(0))A^*]^{-1}\left(\frac{S(\lambda) - S(0)}{\lambda}\right)\eta.$$

When we substitute this value back into (17) we obtain

$$F(z) = \{(S(z) - S(0))\{\eta + A^*[(I-S(\lambda)S(0)^*)-(S(\lambda)-S(0)A^*]^{-1}(S(\lambda)-S(0))\eta\} -$$

$$- (I-S(z)S(0)^*)[(I-S(\lambda)S(0)^*)-(S(\lambda)-S(0))A^*]^{-1}(S(\lambda)-S(0))\eta\}/z-\lambda.$$

Since $D_{Z(A)^*} = D_{Z(A)^*}P_{k_0}$ we have

$$(P_{k_0}F)(z) = (I - S(z)S(0)^*)((- S(0)S(0)^*)^{-1}F(0)$$

$$= (I - S(z)S(0)^*)[(I - S(\lambda)S(0)^*) - (S(\lambda) - S(0))A^*]^{-1}\left(\frac{S(\lambda) - S(0)}{\lambda}\right)\eta$$

and

$$(D_{Z(A)^*}F)(z) =$$

$$(I - S(z)S(0)^*)(I - AA^*)^{\frac{1}{2}}[(I-S(\lambda)S(0)^*)-(S(\lambda) - S(0))A^*]^{-1}\left(\frac{s(\lambda)-S(0)}{\lambda}\right)\eta.$$

Combining this with Definition 2.1 we summarize the results in Theorem 3.2.

THEOREM 3.2. *If $A$ is purely contractive in $N$ and satisfies (7) then $Z(A)$ is a contraction in $M$ whose characteristic function is $\{K_0, k_0, \Theta_{Z(A)}(\lambda)\}$ where $\Theta_{Z(A)}(\lambda)$ is given by*

$$(18) \qquad \Theta_{Z(A)}(\lambda)\left(\frac{S(z) - S(0)}{z}\right)\xi = (I - S(z)S(0)^*)\{- A + (I-AA^*)^{\frac{1}{2}}$$

$$[(I-S(\lambda)S(0)^*)-(S(\lambda)-S(0))A^*]^{-1} \cdot (S(\lambda)- S(0))(I - A^*A)^{\frac{1}{2}}\}\xi.$$

Expression (18) can be brought into a more appealing and clarifying form.

THEOREM 3.3. *If $A$ is purely contractive in $N$ and satisfies (7), then $Z(A)$ is a contraction in $M$ whose characteristic function is $\{K_0, k_0, \Theta_{Z(A)}(\lambda)\}$. $\Theta_{Z(A)}$ is given by*

$$\Theta_{Z(A)}(\lambda)\left(\frac{S(z)-S(0)}{z}\right)\xi = (I - S(z)S(0)^*)G(\lambda)\xi \quad \textit{where} \tag{19}$$

$$(I - S(0)S(0)^*)^{\frac{1}{2}}G(\lambda)(I - S(0)^*S(0))^{-\frac{1}{2}} \tag{20}$$
$$= (I - AA^*)^{\frac{1}{2}}(I - \Gamma(\lambda)A^*)^{-1}(\Gamma(\lambda) - A)(I - A^*A)^{-\frac{1}{2}} \quad \textit{and}$$

$$\Gamma(\lambda) = (I-S(0)S(0)^*)^{\frac{1}{2}}(I-S(\lambda)S(0)^*)^{-1}(S(\lambda)-S(0))(I-S(0)^*S(0))^{-\frac{1}{2}}. \tag{21}$$

Note that we obtain $(I - S(0)S(0)^*)^{\frac{1}{2}}G(\lambda)(I - S(0)^*S(0))^{-\frac{1}{2}}$ as a composition of two matrix fractional linear transformations. For the theory of these transformations, refer to [4].

PROOF. From Theorem 3.2 we have

$$G(\lambda) = -A + (1-AA^*)^{\frac{1}{2}}[(I-S(\lambda)S(0)^*)-(S(\lambda)-S(0))A^*]^{-1}$$
$$(S(\lambda)-S(0))\cdot(I - A^*A)^{\frac{1}{2}}.$$

From the equality $A(A^*A) = (AA^*)A$ we obtain, by familiar arguments,

$$A(I - A^*A)^{\frac{1}{2}} = (I - AA^*)^{\frac{1}{2}}A, \tag{22}$$

$$(I - AA^*)^{-\frac{1}{2}}A = A(I - A^*A)^{-\frac{1}{2}}, \text{ and} \tag{23}$$

$$(I - AA^*)^{-\frac{1}{2}}A(I - A^*A)^{\frac{1}{2}} = A. \tag{24}$$

Using this we obtain

$$G(\lambda) = (I-AA^*)^{-\frac{1}{2}}\{-A + (I-AA^*)\,[I-(I-S(\lambda)S(0)^*)^{-1}(S(\lambda)-S(0))A^*]^{-1}$$
$$\cdot\,(I-S(\lambda)S(0)^*)^{-1}(S(\lambda)-S(0))\}(I - A^*A)^{\frac{1}{2}}$$
$$= (I-AA^*)^{-\frac{1}{2}}\{-A + (I-AA^*)$$
$$[I-(I-S(0)S(0)^*)^{-\frac{1}{2}}\Gamma(\lambda)(I-S(0)^*S(0))^{\frac{1}{2}}A^*]^{-1}$$
$$\cdot\,(I-S(0)S(0)^*)^{-\frac{1}{2}}\Gamma(\lambda)(I-S(0)^*S(0))^{\frac{1}{2}}\}(I-A^*A)^{\frac{1}{2}}.$$

Using (9) and Corollary 2.4 obtain

$$G(\lambda) = (I-S(0)S(0)^*)^{-\frac{1}{2}}(I-AA^*)^{-\frac{1}{2}}\{-A+(I-AA^*)(I-\Gamma(\lambda)A^*)^{-1}\Gamma(\lambda)\}$$
$$\cdot\,(I - A^*A)^{\frac{1}{2}}(I-S(0)^*S(0)^{\frac{1}{2}}$$
$$= (I-S(0)S(0)^*)^{-\frac{1}{2}}(I-AA^*)^{-\frac{1}{2}}(I-AA^*)\,\cdot$$
$$\cdot\,\{-A(I-A^*A)^{-1}+(I-\Gamma(\lambda)A^*)^{-1}\Gamma(\lambda)\}(I-A^*A)^{\frac{1}{2}}(I-S(0)^*S(0))^{\frac{1}{2}}$$

$$= (I-S(0)S(0)^*)^{-\frac{1}{2}}(I-AA^*)^{\frac{1}{2}}(I-\Gamma(\lambda)A^*)^{-1} \cdot$$

$$\cdot \{-(I-\Gamma(\lambda)A^*)A(I-A^*A)^{-1}+\Gamma(\lambda)\}\ (I-A^*A)^{\frac{1}{2}}(I-S(0)^*S(0))^{\frac{1}{2}}.$$

However, using the relations

$$(I-AA^*)^{-1}A = A(I-A^*A)^{-1} \text{ and } I + A^*(I-AA^*)^{-1}A = (I-A^*A)^{-1}$$

it follows that

$$G(\lambda) = (I-S(0)S(0)^*)^{-\frac{1}{2}}(I-AA^*)^{\frac{1}{2}}\,(I-\Gamma(\lambda)A^*)^{-1} \cdot$$

$$\cdot\, (\Gamma(\lambda)-A)\ (I-A^*A)^{-\frac{1}{2}}(I-S(0)^*S(0))^{\frac{1}{2}}$$

which is equivalent to (20).

REMARK. 3.4. Under the assumptions of Theorem 3.3 if

(25) $$S(\lambda)S(0)^*S(0) = S(0)S(0)^*S(\lambda) \text{ for all } \lambda,\ |\lambda| < 1$$

then

(26) $$\Gamma(\lambda) = (I - S(\lambda)S(0)^*)^{-1}(S(\lambda) - S(0)).$$

As corollaries to the above calculations we obtain the following results.

THEOREM 3.5 (i) *Under the assumptions of Theorem* 3.2, $Z(A)^n$, $Z(A)^{*n}$ *converge to zero in the strong operator topology.*

(ii) *The spectrum of* $Z(A)$ *is the union of the set of points* $\lambda$ *on the unit circle where* $S(z)$ *has no analytic continuation and the set of points* $\lambda$ *in the open unit disk at which* $\Gamma(\lambda) - A$ *is not invertible.*

PROOF. $S$ being inner has radial limits almost everywhere which are unitary transformations in $N$. Since fractional linear transformations are continuous and map unitary operators into unitary operators it follows that $\Gamma(\lambda)$ and $(I - S(0)S(0)^*)^{\frac{1}{2}}G(\lambda)(I - S(0)^*S(0))^{-\frac{1}{2}}$ are also inner functions. Thus $\Theta_{Z(A)}(\lambda)$ is inner and hence part (i) follows from [5, Th. VI.2.3].

To prove part (ii), we recall the characterization of the spectrum of a completely non-unitary contraction in terms of its characteristic function $\Theta_T(\lambda)$ [5, Th. VI.4.1]: the spectrum of $T$ is the union of the set of points $\lambda$ on the unit circle where $\Theta_T$ has no analytic continuation and the set of points $\lambda$ in the open unit disk where $\Theta_T(\lambda)$ is not invertible.

Notice that since the fractional linear transformations given by (20) and (21) preserve analyticity, $\Theta_{Z(A)}(\lambda)$ has analytic continuation at exactly the same points of the unit circle where $G(\lambda)$ and $\Gamma(\lambda)$ have continuation. Thus this part of the spectrum of $T$ is stable under the perturbations considered. This is in perfect agreement with Weyl's theorem which states that compact perturbations can only move the point spectrum. As for eigenvalues of $Z(A)$, they are given by points $\lambda$ of the open unit disk at which $G(\lambda)$ is not invertible; by (20), these are the points $\lambda$ where $\Gamma(\lambda) - A$ is not invertible.

The above argument fails for the case where $Z(U)$ is unitary. However [1, Th. 3.2] can still be generalized using Clark's own method.

THEOREM 3.6. *Let $U$ be unitary in $N$, $N$ finite dimensional, and let $U$ satisfy* (7). *Then the spectrum of $Z(U)$ is the union of the set of points $\lambda$ of the unit circle at which $S(z)$ has no analytic continuation and the set of points $\lambda$ of the unit circle at which $S(z)$ has an analytic continuation but $\Gamma(\lambda) - U$ is not invertible.*

PROOF. By Weyl's theorem it suffices to determine the eigenvalues of $Z(U)$ which, since $Z(U)$ is unitary, are the complex conjugates of the eigenvalues of $Z(U)^*$. For the proof it is enough to show that a point $\lambda$ on the unit circle where $S$ has an analytic continuation is an eigenvalue of $Z(U)$ if and only if $\Gamma(\lambda) - U$ is not invertible. Since

$$(Z(U)^* - T^*)F(z) = \left(\frac{S(z) - S(0)}{z}\right)(U^* + S(0)^*)(I - S(0)S(0)^*)^{-1}F(0),$$

solving $Z(U)^*F = \bar{\lambda}F$ reduces to

$$\begin{aligned}(T^* - \bar{\lambda})F(z) &= \left(\frac{S(z) - S(0)}{z}\right)(U^* + S(0)^*)(I - S(0)S(0)^*)^{-1}F(0) \\ &= \left(\frac{S(z) - S(0)}{z}\right)\xi \end{aligned} \tag{27}$$

where $$\xi = (U^* + S(0)^*)(I - S(0)S(0)^*)^{-1}F(0).$$

Now $$(T^* - \bar{\lambda})F(z) = \frac{F(z) - F(0)}{z} - \bar{\lambda}F(z) \text{ and we obtain}$$

$$F(z) = \frac{F(0) + (S(z) - S(0))\xi}{1 - \bar{\lambda}z}.$$

Since $S$ has an analytic continuation at $\lambda$, so have all functions in $M$ [3, p. 76], hence we must have $F(0) = -(S(\lambda) - S(0))\xi$. A simple computation yields

$$(T^* - \bar{\lambda})\left(\frac{S(z) - S(\lambda)}{z - \lambda}\right)\xi = -\bar{\lambda}\left(\frac{S(z) - S(0)}{z}\right)\xi.$$

Substituting back into (27) we obtain

$$\bar{\lambda}\left(\frac{S(z) - S(0)}{z}\right)\{-I + (U^* + S(0)^*)(I - S(0)S(0)^*)^{-1}(S(\lambda) - S(0))\}\xi = 0.$$

From (7) it follows that

$$(U^* + S(0)^*)(I - S(0)S(0)^*)^{-1} = (I - S(0)^*S(0))^{-1}(U^* + S(0)^*).$$

Thus $\lambda$ is an eigenvalue of $Z(U)$ if and only if for some non-zero $\xi \in N$

$$\{-(I - S(0)^*S(0)) + (U^* + S(0)^*)(S(\lambda) - S(0))\}\xi = 0$$

which, since $N$ is finite dimensional, amounts to

$$\{-(I - S(0)^*S(0)) + (U^* + S(0)^*)(S(\lambda) - S(0))\}$$

being non-invertible. But since

$$\{-(I - S(0)^*S(0)) + (U^* + S(0)^*)(S(\lambda) - S(0))\} = U^*\{S(\lambda) - S(0) - U + US(0)^*S(\lambda)\},$$

$\{-(I - S(0)^*S(0)) + (U^* + S(0)^*)(S(\lambda) - S(0)\}$ is invertible if and only if $\{S(\lambda) - S(0) - U + US(0)^*S(\lambda)\}$ is invertible.

Taking adjoints, then multiplying on the left by $S(\lambda)^*$ (which is invertible and unitary wherever $S$ has an analytic continuation), and on the right by $U$ (which is unitary), one sees that $\{S(\lambda) - S(0) - U + US(0)^*S(\lambda)\}$ is invertible if and only if $\{S(\lambda) - S(0) - U + S(\lambda)S(0)^*U\}$ is invertible.

Now we show that $\{S(\lambda) - S(0) - U + S(\lambda)S(0)^*U\}$ is invertible if and only if $\Gamma(\lambda) - U$ is invertible. From (9) we have $U = (I - S(0)S(0)^*)^{\frac{1}{2}}U(I - S(0)^*S(0))^{-\frac{1}{2}}$, hence

$$\Gamma(\lambda) - U = (I - S(0)S(0)^*)^{\frac{1}{2}}\{(I - S(\lambda)S(0)^*)^{-1}(S(\lambda) - S(0)) - U\}(I - S(0)^*S(0))^{-\frac{1}{2}}$$
$$= (I - S(0)S(0)^*)^{\frac{1}{2}}(I - S(\lambda)S(0)^*)^{-1}\{S(\lambda) - S(0) - U - S(\lambda)S(0)^*U\}(I - S(0)^*S(0))^{-\frac{1}{2}}$$

which is invertible if and only if $\{S(\lambda) - S(0) - U - S(\lambda)S(0)^*U\}$ is invertible and the proof is complete.

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